9.1 Intro to Differential Equations

A **differential equation** is an equation involving derivatives.

A **solution to a differential equation** is any function that satisfies the equation.

Entry Task:

Find $y = y(x)$ such that

$$\frac{dy}{dx} - 8x = x^2 \text{ and } y(0) = 5.$$

Check your final answer

$$\frac{dy}{dx} = 8x + x^2$$

$$\Rightarrow \quad y = 4x^2 + \frac{1}{3}x^3 + C$$

$$y(0) = 5 \Rightarrow \quad 5 = 4(0)^2 + \frac{1}{3}(0)^3 + c$$

$$\Rightarrow \quad C = 5$$

$$\boxed{y = 4x^2 + \frac{1}{3}x^3 + 5}$$

CHECK:

- $y(0) = 5$ ✓

$\frac{dy}{dx} = 8x + x^2$

so

- $LHS = \frac{dy}{dx} - 8x = (8x + x^2) - 8x = x^2 = RHS$ ✓

Example

Consider the differential equation:

$$\frac{dP}{dt} = 2P$$

(a) Is $P(t) = 8e^{2t}$ a solution?

$LHS = \frac{dP}{dt} = 16e^{2t}$

$RHS = 2P = 16e^{2t}$

ALWAYS SAME!

$\boxed{\text{YES!}}$

(b) Is $P(t) = t^3$ a solution?

$LHS = \frac{dP}{dt} = 3t^2$

$RHS = 2P = 2t^3$

NOT ALWAYS THE SAME!

$\boxed{\text{No!}}$

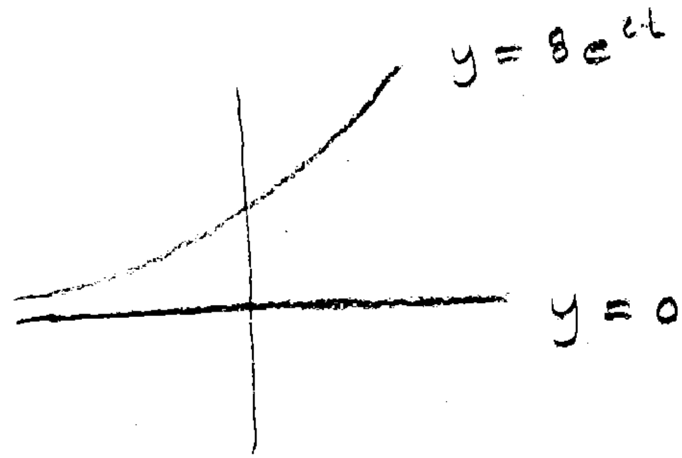
(c) Is $P(t) = 0$ a solution?

$LHS = \frac{dP}{dt} = 0$

$RHS = 2P = 0$

ALWAYS THE SAME

$\boxed{\text{YES}}$



The **general solution** to

$$\frac{dP}{dt} = 2P$$

s

$$P(t) = Ce^{2t},$$

'or any constant C.

*N*e will learn how to find this next time.

Example: Consider the 2nd order differential equation

$$y'' + 2y' + y = 0.$$

(a) Is $y = e^{-2t}$ a solution?

$y' = -2e^{-2t}$, $y'' = 4e^{-2t}$

LHS $= (4e^{-2t}) + 2(-2e^{-2t}) + e^{-2t} = e^{-2t}$

RHS $= 0$ ← NOT THE SAME

$\boxed{\text{NO}}$

(b) Is $y = t e^{-t}$ a solution?

$y' = -te^{-t} + e^{-t} = e^{-t}(1-t)$

$y'' = -e^{-t}(1-t) - e^{-t} = e^{-t}(t-2)$

LHS $= e^{-t}(t-2) + 2e^{-t}(1-t) + te^{-t}$

$= e^{-t}(t - 2 + 2 - 2t + t)$

$= 0$

RHS $= 0$

$\boxed{\text{YES!}}$

(c) There is a sol'n that looks like

$$y = e^{rt}.$$

Can you find the value of *r* that works?

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

LHS $= y'' + 2y' + y$

$= r^2e^{rt} + 2re^{rt} + e^{rt} \stackrel{?}{=} 0$ ← RHS

$e^{rt}(r^2 + 2r + 1) \stackrel{?}{=} 0$

$e^{rt}(r+1)^2 \stackrel{?}{=} 0$

$\boxed{r = -1}$

$\boxed{y = e^{-t}}$ IS A SOL'N.

Application Notes:

$\frac{dy}{dt}$ = "instantaneous **rate of change** of y with respect to t "

"A is proportional to B" means

A = kB, where k is a constant.

In other words, A/B = k.

Ex)

A	B
10	2
100	20
1000	200

$A = 5B$

PROPOTIONALITY CONSTANT

Ex)

P Population of a town	B Babies born in a year
1000	40
10000	400
100,000	4000

$B = kP$

40, 1000

$\Rightarrow k = 0.04$ ← "relative" growth constant

(POPULATION ROUGHLY GROWS AT 4% PER YEAR)

Some examples:

1. Natural Unrestricted population

Assumption: *"The rate of growth of a population is proportional to the size of the population."*

P(t) = the population at year *t*,

$\frac{dP}{dt} =$ the rate of change of the population with respect to time (i.e. rate of growth).

So the assumption is equivalent to

$$\frac{dP}{dt} = kP,$$

for some constant *k*.

2. Newton's Law of Cooling

Assumption: "*The rate of cooling is proportional to the temperature difference between the object and its surroundings.*"

$T_s =$ constant temp. of surroundings

$T(t) =$ temp. of the object at time t,

$\frac{dT}{dt} =$ rate of change of temp. with respect to time (*i.e.* cooling rate).

$T - T_s =$ temp. difference between object and surroundings.

So Newton's Law of Cooling is equivalent to

$$\frac{dT}{dt} = k(T - T_s),$$

for some constant k.

Ex

T Temp of Coffee	T_s Temp of Surroundings	A CHANGE IN TEMP IN ONE MINUTE
170°	70°	−2°
150°	70°	−1.6°
120°	70°	−1°
100°	70°	−0.6°
80°	70°	−0.2°

$B = T - T_s$

$A = k(T - T_s)$

$\uparrow$

-0.02

3. A Mixing Problem

Assume a 50 gallon vat is initially full of pure water.

A salt water mixture is being dumped **into** the vat at 2 gal/min and this mixture contains 3 grams of salt per gal. The vat is thoroughly mixed together.

At the same time, the mixture is coming **out** of the vat at 2 gal/min.

Let y(t) = grams of salt in vat at time *t*.

$\frac{y(t)}{50} =$ salt per gallon in vat at time, *t*.

$\frac{dy}{dt} =$ the rate (g/min) at which salt is changing with respect to time.

Thus,

SALT CONCENTRATION INCOMING MIXTURE

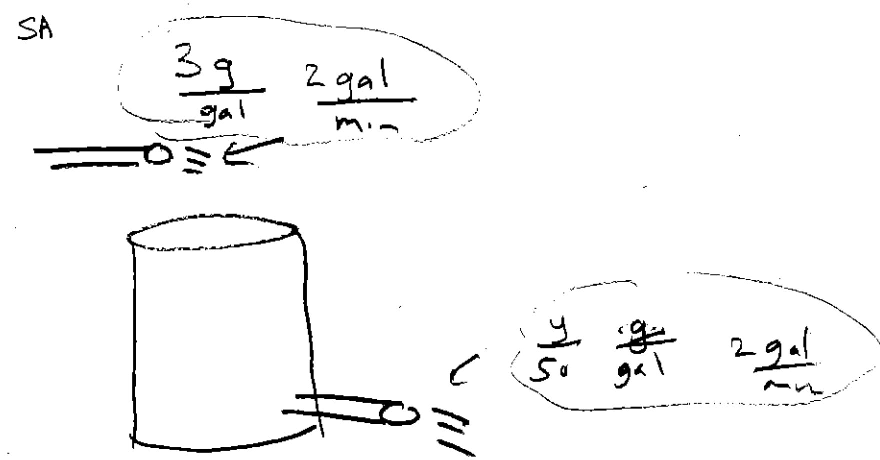
RATE IN = $\left(3\,\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = 6\ \frac{\text{g}}{\text{min}}$

RATE OUT = $\left(\frac{y}{50}\,\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = \frac{y}{25}\ \frac{\text{g}}{\text{min}}$

SALT CONCENTRATION OUTGOING MIXTURE

Thus,

$$\frac{dy}{dt} = 6 - \frac{y}{25}$$



4. All motion problems!

Consider an object of mass m kg moving up and down on a straight line.

Let y(t) = `height at time t'

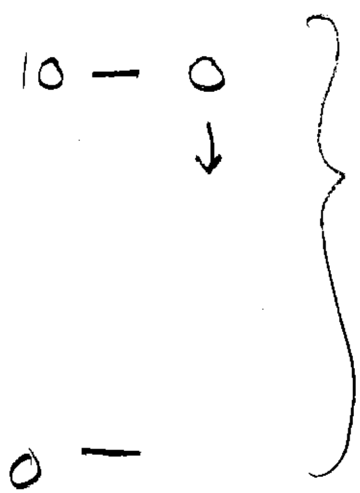
$$\frac{dy}{dt} = \text{`velocity at time } t\text{'}$$

$$\frac{d^2y}{dt^2} = \text{`acceleration at time } t\text{'}$$

Newton's 2nd Law says:

(mass)(acceleration) = Force

$$m\frac{d^2y}{dt^2} = \text{sum of forces on the object}$$



If labeled this way

Only taking into account gravity we get

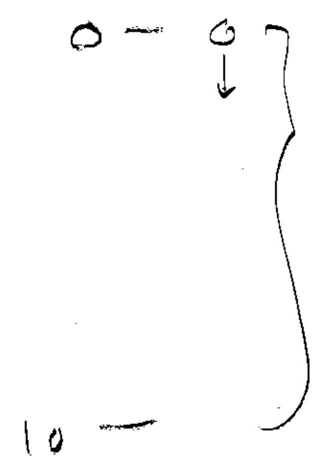
$$m\frac{d^2y}{dt^2} = -mg$$

Now consider gravity and ***air resistance***. One of the most common models is to assume the force due to air resistance is proportional to velocity and in the opposite direction of velocity.

Then we get

ALWAYS A "−" HERE

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}$$



If labeled this way there is a "+" here

5. Many, many others:

Example:

A common assumption for melting snow/ice is "the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area."

Consider a melting snowball:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

Write down the differential equation for *r*.

$$\frac{dV}{dt} = k\,S$$

$$\underbrace{\frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}} = k \cdot \underbrace{4\pi r^2}$$

$$\Rightarrow \quad 4\pi r^2 \frac{dr}{dt} = 4\pi k r^2$$

$$\Rightarrow \quad \boxed{\frac{dr}{dt} = k}$$

RADIUS CHANGING AT A CONSTANT RATE !